Hilbert Space Methods in Hydrodynamics with Applications to Couette Flow

Georg Knorr
Department of Physics and Astronomy, University of Iowa, Iowa City, Iowa 52242

Gerhard Ströhmer
Department of Mathematics, University of Iowa, Iowa City, Iowa 52242

Z. Naturforsch. **48a**, 679–691 (1993); received February 17, 1993

This paper is dedicated to the memory of our friends and colleagues Christoph K. Goertz, Dwight R. Nicholson, Linhua Shan, and Robert A. Smith, who were tragically slain on November 1, 1991

A new base for three-dimensional divergence-free vector fields in Hilbert space (Galerkin method) is proposed. The base fields have vanishing boundary conditions for their curl and are useful to solve the incompressible Navier-Stokes equation. The base vector potentials are obtained as the eigensolutions of the squared Laplace operator. We first derive the operator in a simply connected domain and then study Couette flow in the small gap approximation. The method yields a rapidly converging critical Taylor number and in lowest approximation a three mode model for the Taylor vortices, similar to the Lorenz model. It represents the first bifurcation of the flow very well.

Key words: Hydrodynamics; Eigenvector field; Self-adjoint operator; Couette flow; Taylor vortices.

I. Introduction

The motion of any incompressible Newtonian fluid is governed by the Navier-Stokes equations

$$\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\nabla (p/\varrho) + \nu \nabla^2 \boldsymbol{v}\,, \quad \nabla \cdot \boldsymbol{v} = 0\,. \qquad (1.1)$$

As they apply equally to laminar and turbulent flow, they also ought to be sufficient to derive the various ways of transition to turbulence observed in experiments (for Couette flow, see Di Prima and Swinney [1], for Bénard flow, see Busse [2]).

As the Navier-Stokes equations form a nonlinear system of partial differential equations, their solution is quite difficult, and in three dimensional space even the existence of smooth solutions to smooth initial values is still in doubt (see Leray [3] for the two dimensional case, von Wahl [4] for the three dimensional case). They form an infinite dimensional dynamical system, which has been studied extensively (see Temam [5], in particular Chapter III, 3.2; Chapter IV, 4.2, dealing with fluid flow driven by its boundaries, of which Couette flow is an example). One of the goals of these studies is to find out whether each flow converges to a finite-dimensional manifold, called the inertial manifold, so that the long-term behavior can be described in terms of a finite number of variables. These would be related to the velocities by a nonlinear coordinate system, which would probably be very difficult to obtain.

In contrast to this approach, every finite collection of vectors from a basis, spanning the space of flows, introduces a linear coordinate system in a subspace. It is quite common to project the solutions of the Navier-Stokes equations onto this subspace in order to obtain a system of ordinary differential equations. This was done, e.g., by Hopf [6] in his existence proof for weak solutions of the Navier-Stokes equations. This procedure does not give the exact description of the long term behavior of the flow, which one would obtain from an inertial manifold, but it still gives an approximation of increasing accuracy, as the basis functions sweep the entire space.

The purpose of this paper is to derive a complete set of base flows which are the eigensolutions of a self-adjoint differential operator. By projection of the solutions of the Navier-Stokes equations on these base vectors, we obtain a system of ODEs in the form

$$\partial_t \boldsymbol{X}(t) = \boldsymbol{G}(x)\,, \qquad (1.2)$$

Reprint requests to Prof. Dr. G. Knorr, Department of Physics and Astronomy, University of Iowa, Iowa City, Iowa 52242, USA.

0932-0784 / 93 / 0500-0679 $ 01.30/0. – Please order a reprint rather than making your own copy.

where $\boldsymbol{X}(t)$ represents an infinite-dimensional vector and describes the state of the system.

This is done first in general, then we calculate explicitly Couette flow in the narrow gap approximation.

By truncation (1.2) takes the form of a finite-dimensional dynamical system (see, e.g., Arrowsmith and Place [7], Ruelle [8]). For general numerical methods solving the Navier-Stokes equations see Temam [9] and Girault and Raviart [10]. For numerical methods, especially for Couette flow, see also Mayer-Spasche and Keller [11], Moser et al. [12], Marcus [13, 14], and King et al. [15].

In three dimensional space the transition from (1.1) to a system of the form (1.2) can, e.g., be accomplished by a discrete approximation of the Fourier transform. Many turbulence studies avail themselves of such methods (compare, e.g., Leslie [16]). However, Fourier modes do not take into account a particular geometry. On the other hand, boundary conditions, as encountered, e.g., in Couette flow or Bénard flow, determine the behavior of a flow profoundly, and in particular the transition to turbulence. For nonperiodic boundary conditions one usually takes advantage of special symmetries, introduces stream functions and tries to solve the resulting equations. Much progress has been obtained in this way [17–21, 36] (for the development of the mathematical theory, see Schmitt and von Wahl [22]). This method is quite suitable for the special spatial domains considered and can probably also be adopted to others, but this requires fairly extensive mathematical studies in each case.

The method which is the subject of this paper, evolved from the recent work of Knorr, Lynov, and Pécseli [23], who studied the evolution of the inviscid fluid flow in three dimensional space. However, the simple structure of the base functions and the periodic boundary conditions confines the application of this expansion to a volume which is far removed from any boundaries, where, due to the viscosity of any real fluid, the condition $\boldsymbol{v} = 0$ has to be satisfied.

As Knorr et al. showed [23], it may be beneficial to use the eigenfunctions of another differential operator. The operator of Knorr et al., the curl, is of first order only and therefore cannot be expected to accommodate all the boundary conditions we wish to impose. Thus one would expect that an operator of higher order is necessary to produce the base flows which satisfy the boundary conditions imposed. As we want our base flows to be divergence free, it turns out to be expedient to use an operator of fourth order. Such a basis can also be obtained from the Stokes operator, but the resulting method does not seem to work well, as indicated in the paper of Gebhardt and Grossmann [35]. For other uses of potential functions compare Marqués [36].

In Sect. II we cast the Navier-Stokes equation into a form which best suits our intentions, in Sect. III we derive the operator for a singly connected finite domain. In Sect. IV we formulate the Couette flow in the narrow gap approximation, for which the base vectors are derived in Section V. In Sect. VI we rederive some results of linear theory, using our method of projection on base vectors. It demonstrates the usefulness of our method. In Sect. VII we establish the lowest order nonlinear theory with three modes, the Couette-Lorenz model, and derive some basic results. Section VIII finally summarizes our conclusions.

II. The Navier-Stokes Equations

The incompressible Navier-Stokes equations in a coordinate system rotating with constant angular velocity $\boldsymbol{\Omega}$ are given by

$$\tilde{\varrho}\left(\frac{\partial}{\partial t}\tilde{\boldsymbol{v}}+\tilde{\boldsymbol{v}}\cdot\tilde{\nabla}\tilde{\boldsymbol{v}}\right)=-\tilde{\nabla}\tilde{p}+\mu\tilde{\Delta}\tilde{\boldsymbol{v}}+\varrho\,\tilde{\boldsymbol{v}}\times(2\tilde{\boldsymbol{\Omega}})+\tilde{\boldsymbol{f}}(\boldsymbol{r},t),$$
$$\tilde{\nabla}\cdot\tilde{\boldsymbol{v}}=0;\quad \tilde{\varrho}=\text{const}\,, \qquad (2.1)$$

where the twiddled variables characterize the physical quantities.

The last erm represents an additional force, which may be deterministic (e.g., gravity) or stochastic (e.g., a stirring force). The centrifugal force can be represented by a gradient and has been incorporated into the generalized pressure $\tilde{p}$. We introduce dimensionless variables x, v, t etc. by $\tilde{x}=x_0 x$, $\tilde{v}=v_0 v$, $\tilde{t}=t_0 t$, etc. x_0, v_0, t_0 etc. being dimensional scale factors. The coefficients in (2.1) become all unity by the choice

$$x_0=t_0 v_0,\quad t_0=x_0^2/\nu,\quad v_0=\nu/x_0,$$
$$p_0=\varrho_0\nu^2/x_0^2,\quad \varrho_0=\tilde{\varrho}\,, \qquad (2.2)$$

where $\nu=\mu/\varrho$ is the kinematic viscosity. (2.1) becomes

$$\frac{\partial}{\partial t}\boldsymbol{v}+\boldsymbol{v}\cdot\nabla\boldsymbol{v}=-\nabla p+\Delta\boldsymbol{v}+\Omega\,\boldsymbol{v}\times\boldsymbol{e}_z+\boldsymbol{f}(r,t);\quad \nabla\cdot\boldsymbol{v}=0. \qquad (2.3)$$

$\Delta=\nabla^2$ is the Laplacean operator, $\boldsymbol{e}_z$ is a unit vector in the direction of the rotation axis, and

$$\Omega=2t_0\tilde{\Omega}=2x_0^2\,\tilde{\Omega}/\nu \qquad (2.4)$$

is a dimensionless angular velocity. We also use $\boldsymbol{\Omega} = \Omega \boldsymbol{e}_z$. Note that x_0 is still arbitrary because we have not yet specified any particular geometry. By introducing

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{v} \tag{2.5}$$

and taking the curl of (2.3) we arrive at

$$\frac{\partial}{\partial t} \boldsymbol{\omega} = \nabla \times [\boldsymbol{v} \times (\boldsymbol{\omega} + \boldsymbol{\Omega})] + \Delta \boldsymbol{\omega}, \quad \nabla \cdot \boldsymbol{v} = 0\,. \tag{2.6}$$

This is the vorticity form of the Navier-Stokes equations. It no longer contains the pressure term.

Any vector-field $\boldsymbol{v}(\boldsymbol{r})$ in a simply connected finite space Γ with $\boldsymbol{v}(\boldsymbol{r})=0$ on $\partial\Gamma$ can be represented as follows. First we extend $\boldsymbol{v}(\boldsymbol{r})$ continuously to all of space by setting $\boldsymbol{v}(\boldsymbol{r})=0$ outside Γ. Denoting this new vector field by $\boldsymbol{v}(\boldsymbol{r})$ again, we use the usual Helmholtz decomposition to write

$$\boldsymbol{v} = \nabla \times \boldsymbol{A} - \nabla \Phi. \tag{2.7}$$

As div $\boldsymbol{v}(\boldsymbol{r})=0$ everywhere in the sense of distributions, we get

$$\Delta \Phi = -\nabla \cdot \boldsymbol{v} = 0\,, \tag{2.8}$$

and therefore also $\Phi \equiv 0$. From (2.7) also follows

$$\nabla \times \nabla \times \boldsymbol{A} = \nabla \times \boldsymbol{v} = \boldsymbol{\omega}\,. \tag{2.9}$$

In (2.7) $\boldsymbol{A}$ is only determined up to a gradient, the well known gauge invariance. By solving a Neumann problem in potential theory, we can now replace $\boldsymbol{A}$ by $\boldsymbol{A}' = \boldsymbol{A} + \nabla \Lambda$, with a Λ fulfilling the equations $\Delta\Lambda = -\nabla \cdot \boldsymbol{A}$, $-\boldsymbol{A} \cdot \boldsymbol{n} = (\partial \Lambda/\partial n)$. Then $\boldsymbol{A}' \cdot \boldsymbol{n} = 0$ and $\nabla \cdot \boldsymbol{A}' = 0$. So for any $\boldsymbol{v}$ fulfilling the boundary conditions

$$\boldsymbol{v} = 0 \quad \text{on} \quad \partial\Gamma \tag{2.10}$$

we can find a vector potential $\boldsymbol{A}$ fulfilling the two conditions

$$\nabla \cdot \boldsymbol{A} = 0 \text{ in } \Gamma, \quad \boldsymbol{n} \cdot \boldsymbol{A} = 0 \text{ on } \partial\Gamma. \tag{2.11}$$

Equation (2.9) becomes now

$$\Delta \boldsymbol{A} = -\boldsymbol{\omega}\,, \tag{2.12}$$

and the boundary conditions are

$$\boldsymbol{n} \cdot \nabla \times \boldsymbol{A} = 0; \quad \boldsymbol{n} \times (\nabla \times \boldsymbol{A}) = 0, \quad \boldsymbol{n} \cdot \boldsymbol{A} = 0, \quad \text{on } \partial\Gamma. \tag{2.13}$$

The condition (2.11) on $\boldsymbol{A}$ can be expressed in the following concise way: $\boldsymbol{A}$ always satisfies the condition

$$\int_\Gamma \boldsymbol{A} \cdot \nabla \varphi \, \mathrm{d}\tau = 0\,. \tag{2.14}$$

where φ is a differentiable but otherwise arbitrary function, and $\mathrm{d}\tau$ is the volume element. We denote by H the subspace of L_2 consisting of all vector fields $\boldsymbol{A}$ fulfilling the condition (2.14); it is our basic Hilbert space.

III. The Operator of the Eigenflows

We define the inner product of two vorticity fields $\bar{\boldsymbol{\omega}}$ and $\boldsymbol{\omega}$ by

$$\langle \bar{\boldsymbol{\omega}}, \boldsymbol{\omega} \rangle = \int_\Gamma \bar{\boldsymbol{\omega}}^* \cdot \boldsymbol{\omega} \, \mathrm{d}\tau\,, \tag{3.1}$$

where $\bar{\boldsymbol{\omega}}^*$ is the complex conjugate of $\bar{\boldsymbol{\omega}}$. The integral extends over the domain Γ. We would like to express the inner product (3.1) in terms of the vector potentials $\bar{\boldsymbol{A}}$ and $\boldsymbol{A}$. By partial integration we get

$$\begin{aligned} \langle \bar{\boldsymbol{\omega}}, \boldsymbol{\omega} \rangle &= \int_{\partial\Gamma} \mathrm{d}\boldsymbol{\sigma} \cdot \bar{\boldsymbol{v}}^* \times \boldsymbol{\omega} + \int_\Gamma \mathrm{d}\tau \, \bar{\boldsymbol{v}}^* \cdot \nabla \times \boldsymbol{\omega} \\ &= \langle \bar{\boldsymbol{v}}, \nabla \times \boldsymbol{\omega} \rangle\,. \end{aligned} \tag{3.2}$$

The surface integral vanishes because of (2.10). Starting now with (3.2) and repeating the partial integration, we get

$$\langle \bar{\boldsymbol{\omega}}, \boldsymbol{\omega} \rangle = \int_{\partial\Gamma} \mathrm{d}\boldsymbol{\sigma} \cdot \bar{\boldsymbol{A}}^* \times (\nabla \times \boldsymbol{\omega}) + \langle \bar{\boldsymbol{A}}, \nabla \times \nabla \times \boldsymbol{\omega} \rangle\,. \tag{3.3}$$

We defer the discussion of the surface integral and consider the last term in (3.3), which can be written

$$\langle \bar{\boldsymbol{A}}^*, \nabla \times \nabla \times \nabla \times \nabla \times \boldsymbol{A} \rangle = \langle \bar{\boldsymbol{A}}, \Delta^2 \boldsymbol{A} \rangle \tag{3.4}$$

because of (2.9).

It is now natural to consider the eigenvalue problem

$$\Delta^2 \boldsymbol{A} = \lambda^4 \boldsymbol{A}, \quad \nabla \cdot \boldsymbol{A} = 0 \quad \text{in } \Gamma \tag{3.5}$$

with the boundary conditions $\nabla \times \boldsymbol{A} = 0$, $\boldsymbol{n} \cdot \boldsymbol{A} = 0$ on $\partial\Gamma$.

As we will show, its solutions form a complete set of eigenvectors, so that any divergence free vector field $\boldsymbol{A}$ can be expanded in eigenvectors $\boldsymbol{A}_n$:

$$\boldsymbol{A}(\boldsymbol{r}) = \sum_{n=0}^{\infty} \boldsymbol{A}_n\,. \tag{3.6}$$

For this we must prove that the operator in (3.5) is self-adjoint and not only formally self-adjoint (as defined in Friedman [24]). In other words, we would like to see the surface integral in (3.3) vanish:

$$I = \int_{\partial\Gamma} \mathrm{d}\boldsymbol{\sigma} \, \bar{\boldsymbol{A}}^* \times (\nabla \times \boldsymbol{\omega}) = 0\,. \tag{3.7}$$

However, the integrand does not vanish identically, because $\bar{A}^*$ lies in the surface and $\nabla \times \boldsymbol{\omega}$ has an arbitrary direction.

In order to show that the surface integral vanishes, consider the integral

$$\int_S \mathrm{d}\sigma \cdot \nabla \times \bar{A}^* \tag{3.8}$$

over an arbitrary fraction S of $\partial\Gamma$. Invoking Stokes' theorem, it is equal to the line integral

$$\oint_b \mathrm{d}l \cdot \bar{A}^* , \tag{3.9}$$

where the integral is taken along the closed loop defining S on $\partial\Gamma$. Due to the boundary condition (2.11) both integrals, (3.8) and (3.9), vanish. This means that on $\partial\Gamma$ $\bar{A}^*$ must be a gradient,

$$\bar{A}^* = \nabla\alpha^* . \tag{3.10}$$

Consider next

$$\int_S \mathrm{d}\sigma \cdot \nabla \times (\nabla \times \boldsymbol{\omega}) = \int_S \mathrm{d}\sigma \cdot \nabla \times \nabla \times \nabla \times \nabla \times \boldsymbol{A}$$
$$= \int_S \mathrm{d}\sigma \cdot \Delta^2 \boldsymbol{A} . \tag{3.11}$$

As Δ^2 is to be defined as an operator on H we must incorporate the condition $\Delta^2 A \in H$ into the domain of definition. This implies that

$$\boldsymbol{n} \cdot \Delta^2 \boldsymbol{A} = 0 \quad \text{on } \partial\Gamma,$$

and so

$$\int_S \Delta^2 \boldsymbol{A} \cdot \mathrm{d}\sigma = 0 .$$

As before, we conclude that $\nabla \times \boldsymbol{\omega}$ can be represented on $\partial\Gamma$ by a gradient,

$$\nabla \times \boldsymbol{\omega} = \nabla\beta , \tag{3.12}$$

and that the surface integral in (3.3) becomes now

$$\int_{\partial\Gamma} \mathrm{d}\sigma \cdot \bar{A}^* \times (\nabla \times \boldsymbol{\omega}) = \int_{\partial\Gamma} \mathrm{d}\sigma \cdot \nabla\alpha^* \times \nabla\beta .$$

We continue the functions α^* and β, which so far are only defined on $\partial\Gamma$, into the interior of $\partial\Gamma$. This can be done without difficulty, compare Lighthill [25]. Using Gauss' theorem again, we convert the surface integral into a volume integral:

$$\int_{\partial\Gamma} \mathrm{d}\sigma \cdot \bar{A}^* \times (\nabla \times \boldsymbol{\omega}) = \int_{\Gamma} \mathrm{d}\tau\, \nabla \cdot (\nabla\alpha^* \times \nabla\beta) = 0 . \tag{3.13}$$

We have thus proved that the operator in (3.5) is indeed self-adjoint. Mathematically more precise, the operator possesses a self-adjoint extension by Friedrich's theorem on semi-bounded operators (see, e.g., Yosida [26]). The solutions of (3.5) form a complete set with real eigenvalues, as shown by Ströhmer [27]. All eigensolutions belonging to different eigenvalues are orthogonal to each other. Eigensolutions belonging to the same eigenvalue can be orthogonalized, so that we always have a complete orthogonal set of eigensolutions available. By means of (2.12) we can expand the vorticity $\boldsymbol{\omega}$ in a complete set of orthogonal eigenvectors. Any vorticity field can be written as

$$\boldsymbol{\omega}(\boldsymbol{r}, t) = \sum_{lmn} \alpha_{lmn}(t)\, \boldsymbol{\omega}_{lmn}(\boldsymbol{r}) , \tag{3.14}$$

and the same holds of course for the time derivative. Projecting $\partial_t \boldsymbol{\omega}$ on an arbitrary eigenvector $\boldsymbol{\omega}_{lmn}$ gives

$$\begin{aligned}
\langle \boldsymbol{\omega}_{lmn}, \partial_t \boldsymbol{\omega} \rangle &= \sum_{l'm'n'} \partial_t \alpha_{l'm'n'} \langle \boldsymbol{\omega}_{lmn}(\boldsymbol{r}), \boldsymbol{\omega}_{l'm'n'}(\boldsymbol{r}) \rangle \\
&= \sum_{l'm'n'} \partial_t \alpha_{l'm'n'} \langle \Delta \boldsymbol{A}_{lmn}(\boldsymbol{r}), \Delta \boldsymbol{A}_{l'm'n'}(\boldsymbol{r}) \rangle \\
&= \sum_{l'm'n'} \lambda^4_{l'm'n'}\, \partial_t \alpha_{l'm'n'} \langle \boldsymbol{A}_{lmn}, \boldsymbol{A}_{l'm'n'} \rangle \\
&= \sum_{l'm'n'} \lambda^4_{l'm'n'}\, \partial_t \alpha_{l'm'n'}\, \delta_{ll'}\, \delta_{mm'}\, \delta_{nn'} \\
&= \lambda^4_{lmn}\, \partial_t \alpha_{lmn} .
\end{aligned} \tag{3.15}$$

By our choice of orthogonal base, the Navier-Stokes equations (2.6) can be transformed into a system of first order differential equations in time only by projecting it on the base vectors. The Navier-Stokes equations take on the form

$$\partial_t \alpha_{lmn}(t) = g_{lmn}(\alpha_{ijk}), \quad l, m, n = 0, 1, 2, \ldots . \tag{3.16}$$

The right hand side consists of linear and bilinear terms and has the form

$$g_{lmn}(\alpha_{ijk}) = \sum_{ijk} c^{ijk}_{lmn}\, \alpha_{ijk} + \sum_{ijk} \sum_{rst} c^{ijk,rst}_{lmn}\, \alpha_{ijk}\, \alpha_{rst} .$$

We note that (3.16) is the canonical form in the theory of dynamical systems (1.2) (compare, e.g., Arrowsmith and Place [7], Ruelle [8]), as are (4.1) and (4.2) of Knorr et al. [23].

Rather than projecting $\partial_t \boldsymbol{\omega}(\boldsymbol{r}, t)$ on $\boldsymbol{\omega}_{lmn}$ we can also project it on $\boldsymbol{A}_{lmn}$. We obtain

$$\langle \boldsymbol{A}_{l'm'n'}, \partial_t \boldsymbol{\omega} \rangle = \langle \boldsymbol{v}_{l'm'n'}, \partial_t \boldsymbol{v} \rangle . \tag{3.17}$$

With (3.14) we get

$$\begin{aligned}
\langle \boldsymbol{A}_{l'm'n'}, \partial_t \boldsymbol{\omega} \rangle &= \sum_{lmn} \partial_t \alpha_{lmn} \langle \boldsymbol{v}_{l'm'n'}, \boldsymbol{v}_{lmn} \rangle \\
&= \sum_{lmn} M^{lmn}_{l'm'n'}\, \alpha_{lmn} .
\end{aligned}$$

Ordering the coefficients α_{lmn} into a vector $\boldsymbol{\alpha} = \{\alpha_{lmn}\}$, the Navier-Stokes equation takes on the form

$$\boldsymbol{M}\,\partial_t \boldsymbol{\alpha} = \boldsymbol{G}(\boldsymbol{\alpha}) . \tag{3.18}$$

Multiplication with $\boldsymbol{M}^{-1}$ brings it into the form

$$\partial_t \boldsymbol{\alpha} = \boldsymbol{M}^{-1}\boldsymbol{g}(\boldsymbol{\alpha}) \tag{3.19}$$

which resembles formally (3.16). It is a different system, however, because the base vectors $\boldsymbol{A}_{lmn}$ are different from the $\boldsymbol{\omega}_{lmn}$. Even though there is more work involved in the calculation of (3.19), it might have advantages over (3.16), as we have found.

From (3.17) it is evident that projecting the vorticity on the vectors $\boldsymbol{A}_{lmn}$ is equivalent to projecting the velocity field on the (nonorthogonal) velocity base vectors. The projection of the pressure gradient vanishes always identically. We will apply this method later.

The number of equations in (3.16) or (3.19) is infinite. However, as the eigenvalues of (3.5) grow without bound, the damping term $\Delta\boldsymbol{\omega}$ in (2.6) becomes ever more important, so that we expect a decreasing excitation level of the eigenflows characterized by very large eigenvalues. Nevertheless, the number of excited modes may still be large. We suggest, therefore, to begin the study of (3.19) with a control parameter (e.g., the Reynolds number) which is small so that only few modes are excited. When the control parameter is increased, more and more modes have to be added to the system to maintain a behavior in accordance with reality.

The derivation of (3.16) and (3.19) from (1.1) for a singly connected finite domain is the main result of this paper. In the following sections we will show how the general ideas which led to (3.16) and (3.19) can be modified to be applicable to Couette flow, which is not a flow in a singly connected domain. The following features to be discussed might be noteworthy:

i) The eigenfunctions of (3.5) factor, so that we obtain eigenvalue problems in one variable.
ii) For some components the eigenvector problem of fourth order can be reduced to one of second order.
iii) By the choice of a special coordinate system the eigenflows depend on two variables only, rather than three.
iv) Moving boundaries will be incorporated.

IV. Couette Flow

We consider Couette flow in the narrow gap approximation.

Following Nagata [19], it is evident from Fig. 2 that

$$\tilde{d} = \tilde{r}_2 - \tilde{r}_1 , \quad \bar{r} = \tfrac{1}{2}(\tilde{r}_1 + \tilde{r}_2) , \tag{4.1}$$

where $\tilde{D}$ is the distance between the two cylinders which have the radii $\tilde{r}_1$ and $\tilde{r}_2$. We identify x_0, which was left undetermined in Sect. II, with the gap width $\tilde{d}$. In the narrow gap approximation $\tilde{d}/\bar{r} \ll 1$. A Cartesian coordinate system (in dimensionless coordinates) can be defined by

$$x = \frac{\tilde{r} - \bar{r}}{\tilde{d}} , \quad y = \frac{\bar{r}\,\varphi}{\tilde{d}} , \quad z = \frac{\tilde{z}}{\tilde{d}} , \tag{4.2}$$

where $-1/2 \le x \le +1/2$.

Introducing a rotating coordinate system with

$$\tilde{\Omega} = \tfrac{1}{2}(\tilde{\Omega}_1 + \tilde{\Omega}_2) \tag{4.3}$$

we obtain, according to (2.4),

$$\Omega = \frac{d^2}{v}(\tilde{\Omega}_1 + \tilde{\Omega}_2) . \tag{4.4}$$

In our approximation the Couette flow for small velocities (compare Fig. 1) is given in Cartesian coordinates by the vector potential

$$\boldsymbol{A} = (0, 0, +\tfrac{1}{2}\,\mathfrak{R}\,x^2) , \tag{4.5}$$

from which follows

$$\boldsymbol{V} = (0, -\mathfrak{R}\,x, 0) \tag{4.6}$$

which has a constant vorticity

$$\boldsymbol{W} = \nabla \times \boldsymbol{V} = (0, 0, -\mathfrak{R}) . \tag{4.7}$$

The velocity for the inner cylinder is $V = \bar{r}(\tilde{\Omega}_1 - \tilde{\Omega})/v_0$ at $x = -1/2$, and for the outer cylinder $V = \bar{r}(\tilde{\Omega}_2 - \tilde{\Omega})/v_0$ at $x = +1/2$. Both relations give with (4.3), (2.2) and (4.7)

$$\mathfrak{R} = \frac{(\tilde{\Omega}_1 - \tilde{\Omega}_2)\,\bar{r}\,\tilde{d}}{v} . \tag{4.8}$$

$\mathfrak{R}$ is recognized as the Reynolds number of the narrow gap Couette flow.

Substitute in (2.3) and (2.6)

$$\boldsymbol{v} \to \boldsymbol{v} + \boldsymbol{V} , \quad \boldsymbol{\omega} \to \boldsymbol{\omega} + \boldsymbol{W} , \tag{4.9}$$

where $\boldsymbol{V}$ and $\boldsymbol{W}$ are given by (4.6) and (4.7), to take into account the rotation of the cylinders. The result is

$$\partial_t \boldsymbol{v} - \Delta \boldsymbol{v} = \boldsymbol{v} \times \boldsymbol{\omega} - (\mathfrak{R} - \Omega)\,\boldsymbol{v} + \boldsymbol{e}_z + \mathfrak{R}\,x\,\boldsymbol{\omega} \times \boldsymbol{e}_y - \Delta p , \tag{4.10}$$

$$\partial_t \boldsymbol{\omega} - \Delta \boldsymbol{\omega} = \nabla \times (\boldsymbol{v} \times \boldsymbol{\omega}) - (\mathfrak{R} - \Omega)\,\partial_z \boldsymbol{v} + \mathfrak{R}(x\,\partial_y \boldsymbol{\omega} - \boldsymbol{\omega} \cdot \boldsymbol{e}_x \boldsymbol{e}_y) . \tag{4.11}$$

$\boldsymbol{v}$ satisfies the boundary condition $\boldsymbol{v} = 0$ for $x = \pm 1/2$.

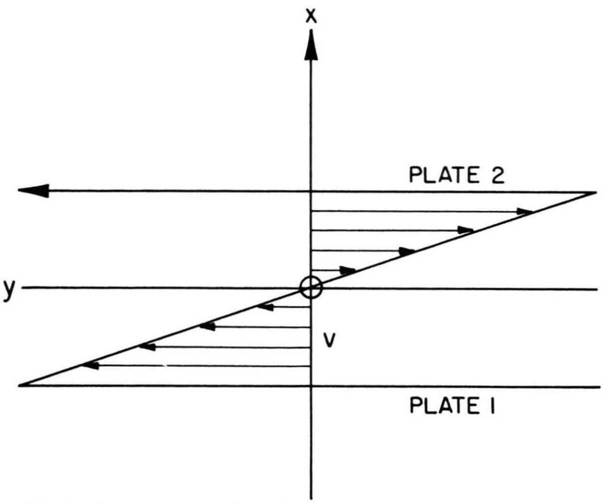

Fig. 1. Couette shear flow between two parallel plates (compare (4.6)).

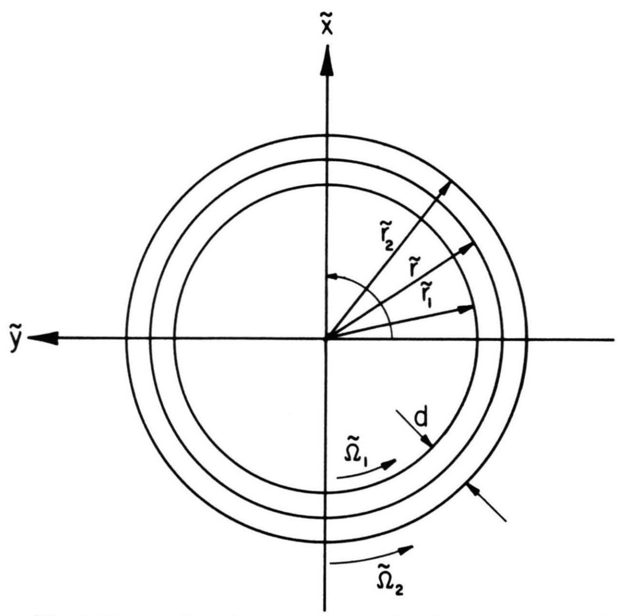

Fig. 2. Couette flow between concentric cylinders. As $d/\bar{r} \to 0$ the geometry of Fig. 1 is approached.

V. Eigenflows between Two Parallel Plates

As the flow between two parallel plates with periodic boundary conditions corresponds to the flow between two tori, our proof in Sect. III is not applicable and we have to analyze this case separately. The geometry is depicted in Figure. 1. The vector potential of the incompressible flow is given by

$$A(x, y, z) = \sum_k A_k = \sum_k a_k(x)\, e^{i k \cdot r}, \tag{5.1}$$

$$k = (0, k_2, k_3)\,. \tag{5.2}$$

It is sufficient to consider one term in the sum. The two preferred directions are the x-axis and k so that it is convenient to introduce a coordinate system with the unit vectors e_x, $\hat{k} \times e_x$, $\hat{k}$ and the coordinates x, η, ζ.

$$\begin{aligned} A_k &= a_k(x)\, e^{i k \cdot r} \\ &= [b_k(x)\, e_x + a_k(x)\, \hat{k} \times e_x + i \hat{k} k^{-1} b'_k(x)]\, e^{ik\zeta}\,. \end{aligned}$$

$a_k(x)$ and $b_k(x)$ are arbitrary functions, independent from each other, and the form of the vector potential has been chosen so as to be divergence free. For convenience we write the variables without index and in the coordinate system defined by e_x, $\hat{k} \times e_x$, $\hat{k}$. We have

$$A_k = [b(x), a(x), i k^{-1} b'(x)]\, e^{ik\zeta}\,, \tag{5.3}$$

$$v_k = \nabla \times A_k = [-i k\, a(x), i k^{-1}(-b'' + k^2 b), a'(x)]\, e^{ik\zeta}, \tag{5.4}$$

$$\omega_k = \nabla \times v_k = [L\, b(x), L\, a(x), i k^{-1} L\, b'(x)]\, e^{ik\zeta}, \tag{5.5}$$

where

$$L = -\,\partial_x^2 + k^2\,. \tag{5.6}$$

On comparing (5.4) and (5.3) with (2.10) and (2.11), the boundary conditions are found to be

$$a(\pm\tfrac{1}{2}) = a'(\pm\tfrac{1}{2}) = 0, \quad b(\pm\tfrac{1}{2}) = 0, \quad L b(\pm\tfrac{1}{2}) = 0\,, \tag{5.7}$$

and the inner product of two vorticity base vectors with the same k is given by

$$\begin{aligned} \langle \bar{\omega}_k, \omega_k \rangle = \int_{-1/2}^{+1/2} [(L\bar{b})(L b) + k^{-2}(L\bar{b}')(L b') \\ + (L\bar{a})(L a)]\, \mathrm{d}x\,. \end{aligned} \tag{5.8}$$

By partial integration and in view of (5.4) and (5.6) we obtain

$$\int_{-1/2}^{1/2} \mathrm{d}x\, \bar{\omega}_{-k} \cdot \omega_k = \int_{1/2}^{1/2} [\bar{a} L^2 a + k^{-2}(L\bar{b})(L^2 b)]\, \mathrm{d}x\,. \tag{5.9}$$

We require

$$L^2 a(x) = \lambda^4 a(x), \qquad a(\pm\tfrac{1}{2}) = a'(\pm\tfrac{1}{2}) = 0; \tag{5.10}$$

$$L^2 b(x) = \sigma^2 L\, b(x), \quad L b(\pm\tfrac{1}{2}) = b(\pm\tfrac{1}{2}) = 0\,. \tag{5.11}$$

The solutions of (5.10) are either even or odd. For the even solutions we put

$$a_\nu(x) = \cos \nu x - \frac{\cos \nu/2}{\cosh \mu/2} \cosh \mu x\,, \tag{5.12}$$

where

$$\mu = \sqrt{\lambda^2 + k^2}, \quad \nu = \sqrt{\lambda^2 - k^2} \tag{5.13}$$

and

$$2\lambda^2 = \mu^2 + \nu^2, \quad \mu^2 = \nu^2 + 2k^2, \quad \nu \le \mu. \tag{5.14}$$

$a(x)$ satisfies automatically $a(\pm 1/2) = 0$. The other boundary condition $a'(\pm 1/2) = 0$ imposes a condition on the eigenvalue λ:

$$\mu \tanh \mu/2 + \nu \tan \nu/2 = 0. \tag{5.15}$$

For $\lambda^2 \gg k^2 \ge 1$ we obtain asymptotically

$$\mu \approx \nu \quad \text{and} \quad \tanh(\mu/2) \approx 1, \tag{5.16}$$

and (5.15) reduces to

$$\tan \nu/2 \approx -1$$

with the solution

$$\nu = (2n + 3/2)\pi. \tag{5.17}$$

If we write (5.15) as

$$\nu = 2(n+1)\pi - 2\tan^{-1}\left\{\frac{\sqrt{\nu^2+2k^2}}{\nu} \tanh \tfrac{1}{2}\sqrt{\nu^2+2k^2}\right\}, \tag{5.18}$$

we recover (5.17) if (5.16) is valid and if for the last term in (5.18) the principal value is chosen. Beginning with $\nu_0 = (2n+3/2)\pi$ and iterating, (5.18) yields ν and, on account of (5.13), the eigenvalue λ^4 rapidly. Note that the eigenvalues depend on k.

For the odd solutions we put

$$a(x) = \sin \nu x - \frac{\sin \nu/2}{\sinh \mu/2} \sinh \mu x \tag{5.19}$$

and the transcendental equation for the eigenvalue becomes $\nu \tanh \mu/2 - \mu \tan \nu/2 = 0$.

In Table 1 are listed some parameters pertaining to the eigenfunctions, and some of the latter are shown in Figure 3.

If we put formally $k=0$ in (5.6), one has $L = -\partial_x^2$ and (5.10) becomes

$$\partial_x^4 a(x) = \lambda^4 a(x). \tag{5.20}$$

This is an equation which has been discussed by Chandrasekhar and Reid [28], Chandrasekhar [29] and others. We remark in passing that we could corroborate the eigenvalues given by Chandrasekhar [29] (1961, page 636, Table LXVIII) through 6 to 8 significant digits.

The eigenvalue problem for the functions $b(x)$, (5.11) becomes simpler than in the general theory. It

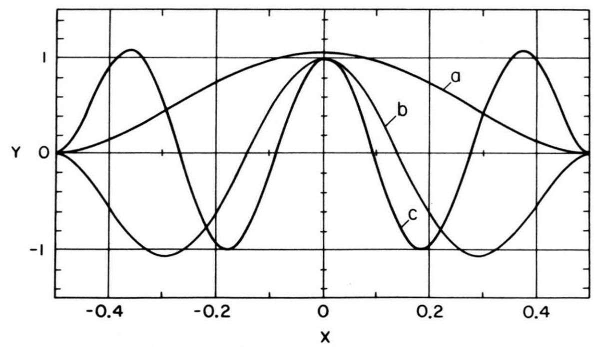

Fig. 3. Some eigenfunctions of (5.10). a) $k=\pi$, $n=0$, $a_{k0}(x)$, b) $k=\pi$, $n=2$, $a_{k2}(x)$, c) $k=\pi$, $n=4$, $a_{k4}(x)$.

Table 1. Some values of ν and λ, related by (5.13) and (5.14) to the eigenvalue λ_{kn}^4. The eigenfunctions are $a_{kn}(x)$ of (5.12) and (5.19).

Order n	k/π	ν	λ
0	0	4.7300409167	4.7300409167
1	0	7.8532047991	7.8532047991
2	0	10.9956081877	10.9956081877
3	0	14.1371658409	14.1371658409
4	0	17.2787601819	17.2787601819
5	0	20.4203527702	20.4203527702
0	1	4.3682087341	5.3805996408
1	1	7.7108632471	8.3262847036
2	1	10.9190631651	11.3620220628
3	1	14.0897879729	14.4357795103
4	1	17.2466391570	17.5304354527
5	1	20.3971779577	20.6376954525
0	2	3.9070007168	7.3988562902
1	2	7.4233997015	9.7254965390
2	2	10.7376577313	12.4408887687
3	3	13.9680723560	15.3161831128
4	2	17.1602676156	18.2743865681
5	2	20.3330763316	21.2817389541

can be reduced to a second order eigenvalue problem

$$L b(x) = \sigma^2 b(x), \quad b(\pm\tfrac{1}{2}) = 0. \tag{5.21}$$

The solutions split into even and odd functions,

$$\text{even:} \quad b_{2n}(x) = \cos \pi(2n+1)x, \qquad \sigma^2 = \pi^2(2n+1)^2 + k^2; \quad n = 0, 1, 2, \ldots,$$

$$\text{odd:} \quad b_{2n-1}(x) = \sin \pi(2n)x, \qquad \sigma^2 = \pi^2(2n)^2 + k^2; \quad n = 1, 2, \ldots. \tag{5.22}$$

The special case $k=0$ describes a flow parallel to the plates. Its vector potential is given by

$$\boldsymbol{A}_0 = d(x)\,\boldsymbol{e}_y + c(x)\,\boldsymbol{e}_z = (0, d(x), c(x)). \tag{5.23}$$

We obtain

$$\boldsymbol{v}_0 = \nabla \times \boldsymbol{A}_0 = (0, -c'(x), d'(x)), \tag{5.24}$$

$$\boldsymbol{\omega}_0 = \nabla \times \boldsymbol{v}_0 = (0, -d''(x), -c''(x)) \tag{5.25}$$

with the boundary conditions $d'(\pm 1/2) = c'(\pm 1/2) = 0$. The inner product of $\bar{\omega}_0$ and ω_0 becomes

$$\langle \bar{\omega}_0, \omega_0 \rangle = \int_{-1/2}^{+1/2} \mathrm{d}x\, [\bar{c}''(x)\, c''(x) + \bar{d}''(x)\, d''(x)] \tag{5.26}$$

$$= -\int_{-1/2}^{+1/2} \mathrm{d}x\, [\bar{c}'(x)\, c'''(x) + \bar{d}'(x)\, d'''(x)],$$

from which we deduce

$$-\partial_x^2 c(x) = \varrho^2 c(x), \quad c'(\pm\tfrac{1}{2}) = 0;$$

$$-\partial_x^2 d(x) = \varrho^2 d(x), \quad d'(\pm\tfrac{1}{2}) = 0. \tag{5.27}$$

The solutions are

$$\text{even:} \quad c_{2m}(x) = d_{2m}(x) = \cos(2m)\pi x, \quad \varrho = 2m\pi, \quad m = 1, 2, \ldots,$$

$$\text{odd:} \quad c_{2m+1}(x) = d_{2m+1}(x) = \sin(2m+1)\pi x, \quad \varrho = (2m+1)\pi, \quad m = 0, 1, 2, \ldots. \tag{5.28}$$

Note that the even zero mode has no physical significance. It should be noted that the eigenfunctions reflect symmetries of the problem, which are crucial for many of the observed phenomena (see Ioss [30], Chaussat et al. [31]).

VI. Couette Flow, Linear Theory

The equation for narrow gap Couette flow is given by (4.10) with $\boldsymbol{v}$ and $\boldsymbol{\omega}$ derived from the vector potential (5.1). We discuss here only the transition from the low velocity shear flow (4.5) to (4.7) to the Taylor vortices. The Taylor flow in lowest order is described by

$$\boldsymbol{A}_k = [b(x), a(x), i k^{-1} b'(x)]\, e^{ikz}, \tag{6.1}$$

resulting in

$$\boldsymbol{v}_k = [-i k\, a(x), i k^{-1} L\, b(x), a'(x)]\, e^{ikz}, \tag{6.2}$$

$$\boldsymbol{\omega}_k = [L\, b(x), L\, a(x), i k^{-1} L\, b'(x)]\, e^{ikz} \tag{6.3}$$

with $L = -\partial_x^2 + k^2$.

We also have to consider in a nonlinear context all smaller numbers than k and must therefore consider a modification of the original shear flow given by

$$\boldsymbol{A}_0 = [0, d(x), c(x)], \tag{6.4}$$

$$\boldsymbol{v}_0 = [0, -c'(x), d'(x)], \tag{6.5}$$

$$\boldsymbol{\omega}_0 = [0, -d''(x), -c''(x)]. \tag{6.6}$$

The real valued $\boldsymbol{A}$ which is used to calculate the physical variables is given by

$$\boldsymbol{A} = \boldsymbol{A}_k + \boldsymbol{A}_{-k} + \boldsymbol{A}_0.$$

The modes derived from $\boldsymbol{A}_k$ and $\boldsymbol{A}_0$ are the smallest number of modes which can give us useful physical information about Couette flow. Before going into the nonlinear regime it is instructive to compare our results with those available in the literature in the well established realm of linear theory. We neglect the nonlinear term $\nabla \times (\boldsymbol{v} \times \boldsymbol{\omega})$ in (4.11), which then reduces to

$$\partial_t \boldsymbol{\omega} = (\Omega - \mathfrak{R})\, \partial_z \boldsymbol{v} + \mathfrak{R}(x\, \partial_y \boldsymbol{\omega} - \boldsymbol{\omega} \cdot \boldsymbol{e}_x\, \boldsymbol{e}_y) + \Delta \boldsymbol{\omega}. \tag{6.7}$$

Inserting the flow resulting from $\boldsymbol{A}_k$ gives for the x, y, and z components of (6.7)

$$\partial_t L\, b(x) = (\Omega - \mathfrak{R})\, k^2\, a(x) - L^2\, b(x), \tag{6.8}$$

$$\partial_t L\, a(x) = -\Omega\, L\, b(x) - L^2\, a(x), \tag{6.9}$$

$$\partial_t i k^{-1} L\, b'(x) = (\Omega - \mathfrak{R})\, i k\, a'(x) - i k^{-1} L^2\, b'(x), \tag{6.10}$$

of which only (6.8) and (6.9) are linearly independent.

The flow resulting from $\boldsymbol{A}_0$ gives for the y and z components

$$\partial_t(\partial_x^2 c(x)) = +\partial_x^4 c(x). \tag{6.11}$$

These components are always damped and can be neglected.

Multiplying (6.8) with Ω, introducing the Taylor number

$$T = \Omega(\mathfrak{R} - \Omega)$$

and substituting $b(x)$ for $\Omega\, b(x)$ puts (6.8) and (6.9) into the form

$$\partial_t L\, b = -(T k^2 a + L^2 b),$$

$$\partial_t L\, a = -(L\, b + L^2 a). \tag{6.12}$$

Due to the lack of an explicit time dependence, we assume a time variation of the form $\exp(\lambda t)$. The case of marginal stability is given by $\mathrm{Re}(\lambda) = 0$. As postulated by Serrin [32] and proved by Yih [33] the imaginary part of λ then also has to vanish. Then we obtain

$$T k^2\, a(x) + L^2\, b(x) = 0,$$

$$L\, b(x) + L^2\, a(x) = 0. \tag{6.13}$$

Solving for $a(x)$ gives

$$L^3 a(x) = T k^2 a(x), \quad -\tfrac{1}{2} \leq x \leq +\tfrac{1}{2}. \tag{6.14}$$

The boundary conditions are obtained from (6.2) and (6.13):

$$a(\pm\tfrac{1}{2}) = a'(\pm\tfrac{1}{2}) = L^2 a(\pm\tfrac{1}{2}) = 0.$$

Equation (6.14) is an eigenvalue equation for T and has been solved by Reid and Harris [34]. T is a function of k which in planar infinite geometry may take any value. Nature chooses a k which makes T a minimum, i.e., the flow bifurcates into Taylor vortices as soon as there is an unstable k. For the minimum Reid and Harris give

$$T = 1707.762 \quad \text{for a wave number of } k = 3.117. \tag{6.15}$$

These values are also quoted in Chandrasekhar's book [29] (1961, p. 43 and p. 310). The minimum k deviates from π by less than 1%, giving rise to vortices in what is almost a square box.

In our approach we expand $a(x)$ and $b(x)$ in (6.8) to (6.10) in the appropriate eigenfunctions $a_{k\nu}(x)$ and $b_{k\nu}(x)$, given by (5.12), (5.19), and (5.22), respectively.

$$a(x) = \sum_\nu \alpha_{k\nu}(t)\, a_{k\nu}(x),$$

$$b(x) = \sum_\nu \beta_{k\nu}(t)\, b_{k\nu}(x). \tag{6.16}$$

The Eqs. (6.12) are first projected on the eigenmode of the vector potential,

$$[b_{k\nu}, 0, i k^{-1} b_{k\nu}], \tag{6.17}$$

resulting in

$$\partial_t \langle L^2 b_{k\nu}, b\rangle = -T k^2 \langle L b_{k\nu}, a\rangle - \langle L^3 b_{k\nu}, b\rangle.$$

Making use of the eigenvalue Eq. (5.21) results in

$$\sigma_{k\nu}^2\, \partial_t \langle b_{k\nu}, b\rangle = -T k^2 \langle b_{k\nu}, a\rangle - \sigma_{k\nu}^4 \langle b_{k\nu}, b\rangle, \quad \nu = 0, 1, 2, \ldots.$$

For $\partial_t = 0$ we have

$$T k^2 \langle b_{k\nu}, a\rangle + \sigma_{k\nu}^4 \langle b_{k\nu}, b\rangle = 0, \tag{6.18}$$

where a and b are given by (6.16).

Projecting (6.12) on the vectorpotential

$$[0, a_{k\nu}(x), 0] \tag{6.19}$$

we obtain

$$\partial_t \langle L a_{k\nu}, L a\rangle = -\{\langle L a_{k\nu}, L b\rangle + \langle L a_{k\nu}, L^2 a\rangle\}. \tag{6.20}$$

The time independent state becomes, using (5.10),

$$\langle L a_{k\nu}, b\rangle + \lambda_{k\nu}^4 \langle a_{k\nu}, a\rangle = 0. \tag{6.21}$$

The Eqs. (6.21) and (6.18), together with (6.16), are equivalent to (6.13). Inserting (6.16), and eliminating the α_ν leads to a standard eigenvalue problem in linear algebra,

$$(A_{in} - \Lambda \delta_{in})\, \beta_n = 0,$$

where

$$A_{im} = \sum_{j=0}^{\infty} (\langle b_i, a_j\rangle \langle a_j, b_n\rangle\, \delta_n^2 / \sigma_i^4 \lambda_j^4 \,\|a_j\|^2 \,\|b_j\|^2)$$

and $\Lambda = (T k^2)^{-1}$.

If we take only one term of the matrix A_{ik}, namely A_{00}, we get

$$T = T_1 = \frac{\sigma_{k0}^2\, \lambda_{k0}^4\, \|a_{k0}\|^2\, \|b_{k0}\|^2}{\langle b_{k0}, a_{k0}\rangle^2\, k^2}. \tag{6.22}$$

The result for $k = \pi$ is $T_1 = 1716.96$, which deviates from the correct result (6.15) only by 1/2%.

In the approach described, we projected (6.12) on base vectors of the vector potential. Now we apply the first method of Sect. III and project it on base vectors of the vorticity. Taking again the lowest approximation, we obtain a different expression for T, namely

$$T = T_2 = \frac{\lambda_{k0}^4\, \sigma_{k0}^2\, \|a_{k0}\|^2\, \|b_{k0}\|^2}{k^2 \langle a_{k0}, b_{k0}\rangle^2} \; \frac{\langle L a_{k0}, a_{k0}\rangle}{\sigma_{k0}^2\, \|a_{k0}\|^2}$$

$$= T_1 \frac{\langle L a_{k0}, a_{k0}\rangle}{\sigma_{k0}^2\, \|a_{k0}\|^2}.$$

Inserting numerical values for $k = \pi$ we find $(\langle L a_0, a_0\rangle / \sigma_0^2 \,\|a_0\|^2) = 1.110$.

Results for larger numbers of functions are given in Table 2 for both methods. The selection of these modes was done by symmetry considerations. The first is called the v-method, the other the ω-method. The convergence of the former is much more rapid, as one easily sees. From the results we conclude that the minimum critical Taylor number has a value $T_c = 1707.76178$ for a wave number $k = 3.11632$, two more digits than in the paper of Reid and Harris [34].

VII. Couette Flow, Nonlinear Theory

The damping decrement of the viscous term in the NS equation increases with the eigenvalue of a particular mode. It is thus meaningful to order the eigenmodes according to the magnitude of their damping decrements and to truncate the infinite set at a particular value. As the order of the eigenfunction operators

Table 2a. Critical Taylor number with the v-method. In the v-method the conditions for marginal stability as derived from the Navier-Stokes equation are projected on a subspace of base velocity vectors with dimension n. It is seen that the convergence is quite good.

No. of modes	k	T_n
2	3.11371	1716.770267
4	3.11619	1707.982699
8	3.11632	1707.764651
16	3.11632	1707.761804
32	3.11632	1707.761777

Consecutive differences and their ratios (approximate)

$D_1 = T_2 - T_4 = 8.79, \quad D_3 = T_8 - T_{16} = 0.00285,$

$D_2 = T_4 - T_8 = 0.218, \quad D_4 = T_{16} - T_{32} = 0.000027,$

$D_1/D_2 = 40, \quad D_2/D_3 = 77, \quad D_3/D_4 = 105.$

Tabelle 2b. Critical Taylor number with the ω-method. In the ω-method the stability conditions as they follow from the vorticity equation are projected on a subspace of dimension n spanned by vorticity base vectors. The convergence with this method is slow as compared with the v-method.

No. of modes	k	T
2	3.24357	1903.307363
4	3.18407	1803.488777
8	3.15160	1757.031356
16	3.13430	1732.624455
32	3.12540	1720.207527

Consecutive differences and their ratios (approximate)

$D_1 = T_2 - T_4 = 99.8, \quad D_3 = T_8 - T_{16} = 24.4,$

$D_2 = T_4 - T_8 = 46.5, \quad D_4 = T_{16} - T_{32} = 12.4,$

$D_1/D_2 = 2.2, \quad D_2/D_3 = 1.9, \quad D_3/D_4 = 2.0.$

varies, we choose the following quantities, which are roughly equivalent to the damping decrement of a given mode:

$$\begin{aligned} \lambda_{k\nu}^2/\pi^2 &\simeq (\nu + 3/2)^2 + k^2/\pi^2 && \text{for } a_{k\nu}(x), \\ \sigma_{k\nu}^2/\pi^2 &= (\nu+1)^2 + k^2/\pi^2 && \text{for } b_{k\nu}(x), \\ \varrho_\nu^2/\pi^2 &= \nu^2 && \text{for } c_\nu(x) \text{ and } d_\nu(x). \end{aligned} \tag{7.1}$$

The quoted value for $\lambda_{k\nu}^2$ is asymptotic for large ν and therefore only approximate.

The magnitude of the damping decrements says little about the amplitude of the corresponding eigenmode because the amplitude depends also on the level of excitation. The numerical experiment will decide, which modes have to be taken into account, given a certain excitation level.

The modes c_1 and d_1 do not interact with other modes and are always damped so that their amplitude must vanish. The mode $d_2(x)$ cannot exist for a cylinder of finite height. The only surviving modes are $b_{k0}(x)$, $a_{k0}(x)$ and $c_2(x)$. For $k=\pi$ one obtains

$$\begin{aligned} \boldsymbol{A}_k(\boldsymbol{r}, t) &= [\beta(t)\, b_{k0}(x), \alpha(t)\, a_{k0}(x), i\,k^{-1}\beta(t)\, b'_{k0}(x)]\, e^{ikz}, \\ \boldsymbol{v}_k(\boldsymbol{r}, t) &= [-i\,k\,\alpha(t)\, a_{k0}(x), i\,k^{-1}\beta(t)\, L\, b_{k0}(x), \alpha(t)\, a'_{k0}(x)]\, e^{ikz}, \\ \boldsymbol{\omega}_k(\boldsymbol{r}, t) &= [\beta(t)\, L\, b_{k0}(x), \alpha(t)\, L\, a_{k0}(x), i\,k^{-1}\beta(t)\, L\, b'_{k0}(x)]\, e^{ikz} \end{aligned} \tag{7.2}$$

and

$$\begin{aligned} \boldsymbol{A}_0(x, t) &= [0, 0, \gamma(t)\, c_2(x)]\,, \\ \boldsymbol{v}_0(x, t) &= [0, -\gamma(t)\, c'_2(x), 0]\,, \\ \boldsymbol{\omega}_0(x, t) &= [0, 0, -\gamma(t)\, c''_2(x)]\,, \end{aligned} \tag{7.3}$$

which is similar to (6.1) to (6.6). $\alpha(t)$ and $\beta(t)$ are in general complex numbers, whereas $\gamma(t)$ is always real. The most general flow with these modes is given by

$$\boldsymbol{v} = \boldsymbol{v}_k + \boldsymbol{v}_0 + \boldsymbol{v}_{-k}\,, \quad \boldsymbol{\omega} = \boldsymbol{\omega}_k + \boldsymbol{\omega}_0 + \boldsymbol{\omega}_{-k}\,, \tag{7.4}$$

where $\boldsymbol{v}_{-k} = \boldsymbol{v}_k^*$ and $\boldsymbol{\omega}_{-k} = \boldsymbol{\omega}_k^*$.

Equation (7.4) is inserted into (4.10) and projected on the three linearly independent velocity vectors

$$\boldsymbol{v}_k^1(x, z) = [-i\,k\, a_{k0}(x), 0, a'_{k0}(x)]\, e^{ikz}\,, \tag{7.5}$$

$$\boldsymbol{v}_k^2(x, z) = [0, i\,k^{-1}\, L\, b_{k0}(x), 0]\, e^{ikz}\,, \tag{7.6}$$

$$\boldsymbol{v}_0^1(x) = [0, c'_2(x), 0]\,. \tag{7.7}$$

Replacing β by $\Omega\beta$ and γ by $\Omega\gamma$, the resulting equations take the form

$$\begin{aligned} \partial_t \alpha &= -A_1 \alpha - A_2 \beta\,, \\ \partial_t \beta &= -T B_1 \alpha - B_2 \beta + B_{13}\alpha\gamma\,, \\ \partial_t \gamma &= -C_3 \gamma - C_{12}\, \tfrac{1}{2}(\alpha\beta^* + \alpha^*\beta)\,. \end{aligned} \tag{7.8}$$

The coefficients are all positive and real, and are given by

$$A_1 = \frac{\lambda_{k0}^4 \langle a_{k0}, a_{k0}\rangle}{\langle L\, a_{k0}, a_{k0}\rangle} = 38.24646\,,$$

$$A_2 = \frac{\sigma_{k0}^2 \langle a_{k0}, b_{k0}\rangle}{\langle L\, a_{k0}, a_{k0}\rangle} = 0.94181\,,$$

$$B_1 = \frac{k^2 \langle b_{k0}, a_{k0}\rangle}{\sigma_{k0}^2 \langle b_{k0}, b_{k0}\rangle} = 0.46687\,,$$

$$B_2 = \sigma_{k0}^2 = 19.7392\,,$$

$$B_{13} = \frac{\varrho_2^2 \, k^2 \langle b_{k0}, a_{k0} \, c_2 \rangle}{\sigma_{k0}^2 \langle b_{k0}, b_{k0} \rangle} = 10.60849 \,,$$

$$C_3 = \varrho_2^2 = 39.478418 \,, \tag{7.9}$$

$$C_{12} = \frac{2 \sigma_{k0}^2 \langle c_2, a_{k0} \, L \, b_{k0} \rangle}{\langle c_2, c_2 \rangle} = 2 B_{13} = 21.21698 \,.$$

According to our derivation the variables $\alpha(t)$ and $\beta(t)$ are complex while $\gamma(t)$ is real. If α and β are chosen real at $t=0$ they remain so for all times:

$$\begin{aligned} \partial_t \alpha &= -A_1 \alpha - A_2 \beta \,, \\ \partial_t \beta &= -T B_1 \alpha - B_2 \beta + B_{13} \alpha \gamma \,, \\ \partial_t \gamma &= -C_3 \gamma - C_{12} \alpha \beta \,. \end{aligned} \tag{7.10}$$

We note that (7.9) has the same structure as in the Lorenz model [37], which is usually written as

$$\begin{aligned} \dot{X} &= \sigma(-X+Y) \,, \\ \dot{Y} &= r X - Y - XZ \,, \\ \dot{Z} &= -bZ + XY \,. \end{aligned} \tag{7.11}$$

However, the numerical values of the coefficients for (7.10) are different from those chosen by Lorenz.

We first study the singular points for (7.8). For $\partial_t = 0$ we have

$$\begin{aligned} A_1 \alpha_0 + A_2 \beta_0 &= 0 \,, \\ T B_1 \alpha_0 + B_2 \beta_0 - B_{13} \alpha_0 \gamma_0 &= 0 \,, \\ C_3 \gamma_0 + C_{12} \alpha_0 \beta_0 &= 0 \,. \end{aligned}$$

For these we find two solutions:

$$\alpha_0 = \beta_0 = \gamma_0 = 0$$

and

$$\begin{aligned} \alpha_0 &= \pm h \sqrt{T/T_c - 1} \,, \\ \beta_0 &= -\frac{A_1}{A_2} \alpha_0 \,, \\ \gamma_0 &= \frac{A_1 C_{12}}{A_2 C_3} \alpha_0^2 \,, \end{aligned} \tag{7.12}$$

where

$$\begin{aligned} T_c &= (A_1 B_2 / A_2 B_1) \\ &= (\lambda_{k0}^4 \, \sigma_0^2 \, \| a_{k0} \| \, \| b_{k0} \| / k^2 \langle a_{k0}, b_{k0} \rangle^2) = 1717 \,, \end{aligned}$$

and

$$\begin{aligned} h &= \sqrt{(C_3 B_2 / C_{12} B_{13})} \\ &= (\sigma_0 \, \| c_2 \| \, \| b_{k0} \| / \sqrt{2} k \, \langle a_{k0} \, b_{k0}, c_2 \rangle) = 1.61 \,. \end{aligned}$$

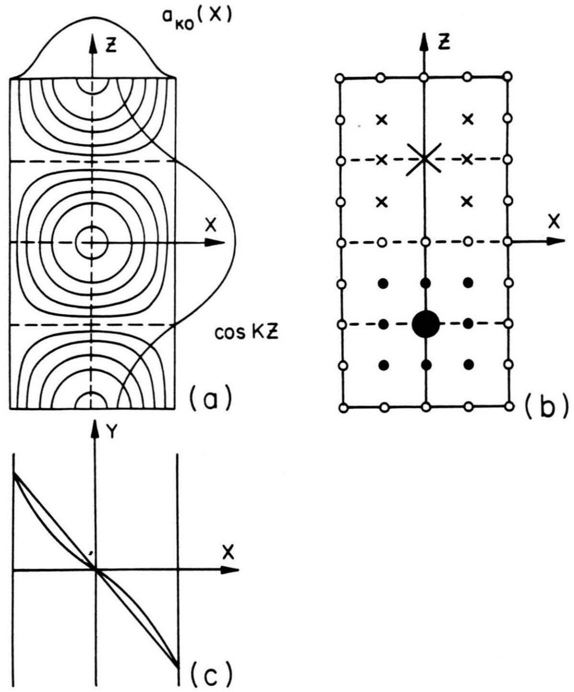

Fig. 4. Taylor vortex flow, as represented by (7.10) (schematic). a) $\underline{v}_a = \boldsymbol{e}_y \times \nabla \boldsymbol{\Phi}$ with $\boldsymbol{\Phi} = \boldsymbol{\Phi}_0 \, a_{k0}(x) \cos k z$. The contour lines of $\boldsymbol{\Phi}(x, z)$ are the stream lines. b) The parallel flow (7.14) depending on x and z. The flow is always into or out of the paper plane. • is flow out of, × is flow into the paper plane. o signifies no flow. The magnitude of the symbol indicates roughly the magnitude of the flow. c) Parallel flow (7.15) depending on x only. It is a modification of the original Couette flow (4.6).

The value of T_c is the same as discussed in the preceding section. As long as $T < T_c$ the amplitude α is imaginary and only the solution $\alpha_0 = \beta_0 = \gamma_0 = 0$ is physically meaningful. A linear stability analysis shows that it is stable for $T < T_c$. In other words, only the shear flow (4.6), (4.7) is possible. As soon as $T \geq T_c$ the solution $\alpha_0 = \beta_0 = \gamma_0 = 0$ becomes unstable and the system goes over to one of the two fixed points of (7.12). The Taylor vortices emerge. The diagram in Fig. 5 shows that it is a pitchfork bifurcation. α_0 and β_0 can be positive or negative because adjacent Taylor vortices have different flow directions. Changing the sign of α_0 corresponds to a shift along the z-axis by one half of the periodicity length.

The geometrical significance of the modes $\alpha(t)$, $\beta(t)$, $\gamma(t)$ is the following: $\alpha(t)$ describes the Taylor vortices. This is most easily seen by putting β to zero in the equation for $\boldsymbol{v}_k$ in (7.2), adding the complex conjugate and dividing by two. The resulting real velocity can be

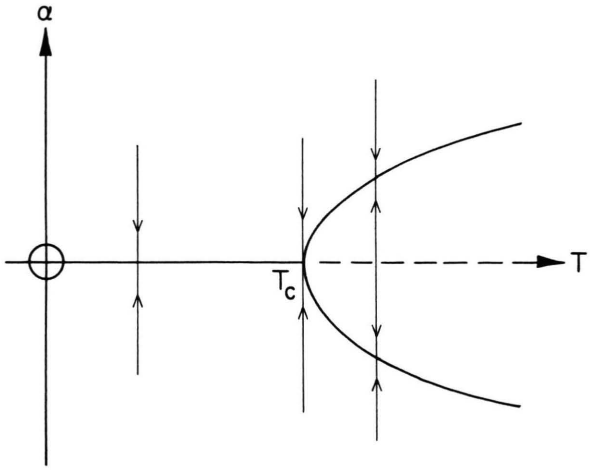

Fig. 5. Pitchfork Bifurcation diagram for Taylor vortices at $T = T_c$. The thin arrows indicate growth or decay of the amplitude α.

written as

$$\boldsymbol{v}_a = \boldsymbol{e}_y \times \nabla \boldsymbol{\Phi}\,, \quad \text{where } \boldsymbol{\Phi} \text{ is given by}$$

$$\boldsymbol{\Phi}(x, z) = \boldsymbol{\Phi}_0\, a_{k0}(x) \cos kz\,, \tag{7.13}$$

with $a_{k0}(x)$ given by (5.12). The equipotential lines of $\boldsymbol{\Phi}(x, z)$ are the stream lines. They are plotted schematically in Figure 4a. They correspond to the Taylor [38] vortices.

The Taylor vortices are accompanied by

$$\boldsymbol{v}_b = [0, -\cos \pi x \sin kz, 0]\,, \tag{7.14}$$

which is shown in Figure 4b. The o indicates zero velocity, × is a flow into and • a flow out of the paper plane. The intensity is roughly indicated by the size of the symbol.

The third flow, without z-dependence, is obtained from (7.3):

$$\boldsymbol{v}_0 = [0, -c_2'(x), 0] \cong [0, \sin 2\pi x, 0]\,. \tag{7.15}$$

It represents a modification of the Couette shear flow (4.6) and has, to our knowledge, first been discussed by Nagata [19]. Both flows are shown in Figure 4c. The modification tends to increase the gradients near the walls and decrease them in the interior.

This modification is necessary to extend the validity of the Taylor vortex flow into the nonlinear domain for $T > T_c$. If we omit this mode the system becomes linear. Only for $\gamma(t) \neq 0$ the solutions with Taylor vortices can be extended into the nonlinear domain of $T > T_c$.

We investigate the linear stability of the Taylor vortices (7.12) by writing in (7.9) $\alpha = \alpha_0 + \alpha'$, $\beta = \beta_0 + \beta'$, $\gamma = \gamma_0 + \gamma'$ and keeping only linear terms. The resulting system becomes

$$\begin{aligned} \dot{\alpha}' &= -A_1 \alpha' - A_2 \beta'\,, \\ \dot{\beta}' &= -\frac{A_1 B_2}{A_2} \alpha' - B_2 \beta' + B_{13} \alpha_0 \gamma'\,, \\ \dot{\gamma}' &= C_{12} \frac{A_1}{A_2} \alpha_0 \alpha' - C_{12} \alpha_0 \beta' - C_3 \gamma'\,. \end{aligned} \tag{7.17}$$

It contains the amplitude of the Taylor vortex α_0 as parameter, which is, via (7.12) directly related to the Taylor number T. With $\alpha'(t) = \alpha' \exp(\lambda t)$ etc. we obtain a cubic polynomial for λ which for small α_0 has only negative solutions for α_0. If T grows with α_0 beyond

$$\frac{T}{T_c} = 1.34\,, \tag{7.19}$$

(T_c = critical Taylor number) two solutions λ become complex, but all λs have a negative real part. Thus the vortices of the 3-mode system (7.12) are stable for $1 < T/T_c < \infty$. This rersult is an indication for the amazing persistence of perturbed Taylor vortices for large T, which is observed experimentally. Fenstermacher et al. [39] for example produced a turbulent flow in their Couette experiment. Nevertheless, the axial periodicity as described by the Taylor vortices persisted.

VIII. Summary and Conclusions

In the first part of thie paper we decompose an arbitrary divergence-free vector potential which describes an incompressible hydrodynamic flow in a simply connected finite domain into orthogonal divergence-free base vector fields such that the boundary condition $v = 0$ is automatically satisfied. This is accomplished by a linear self-adjoint operator of fourth order which defines an eigenvalue problem. The solution vector fields form a complete set in Hilbert space. A general velocity field is obtained by a linear superposition of base vector fields. The Navier-Stokes equation, when projected on the base vector fields gives a set of first order ODEs in time of the form

$$M_{ji}\, \partial_t X_i(t) = G_j(X_k)$$

(the v-method), which can be diagonalized. An alternative way is to project the vorticity form of Navier-Stokes on eigenfields of the vorticity, which yields immediately an orthogonal system of the form

$$\partial_t X_i(t) = G_j(X_k),$$

(the ω-method).

In the second part of the paper we investigate explicitly Couette flow in the narrow gap approximation. The boundary conditions for rotating cylinders are incorporated. The standard results of linear theory are reproduced with the v-method with three eigenmodes to 1/2% and with 32 eigenmodes to 9 significant digits. The ω-method gives slower convergence. Taking three modes for the dynamical case, we find a nonlinear system of the structure of the Lorenz system, but with different numerical coefficients. With increasing Taylor number we find that shear Couette flow is stable for $T < T_c$. For $T = T_c$ the flow bifurcates into Taylor vortices (pitch fork bifurcation). One flow represents the Taylor vortices, another is in φ- or y-direction and depends on x and z. A third flow modifies the original shear flow. These three flows stabilize themselves nonlinearly such that the amplitude of the Taylor vortices is given by

$$\alpha_0 \sim \sqrt{T/T_c - 1}\,.$$

Acknowledgements

The authors thank their students, I. Christopher, T. Liu, and T. Nakamura, for assistance and encouragement by their enthusiasm. The Center of Advanced Studies and the Department of Physics and Astronomy of the University of Iowa are acknowledged for summer stipends.

One of us (G.K.) acknowledges research visits to the Danish National Laboratory at Risø and the Max-Planck Institute for Plasma Physics at Munich. Numerous discussions with D. Biskamp, F. Busse, N. D'Angelo, P. Graeff, J. P. Lynov, D. Nicholson, H. Pécseli, D. Pfirsch, J. J. Rasmussen, H. P. Schamel, K. Schindler, S. Spangler, S. Traugott, and H. Weitzner are gratefully acknowledged.

[1] R. C. Di Prima and H. L. Swinney, in: Hydrodynamic Instabilities and the Transition to Turbulence (H. L. Swinney and J. P. Gollup, eds.), 1981, Chapter 6.
[2] F. H. Busse, in: Hydrodynamic Instabilities and the Transition of Turbulence (H. L. Swinney and J. P. Gollup, eds.), 1981, Chapter 5.
[3] J. Leray, Acta Math. **63**, 193 (1934).
[4] W. Von Wahl, The Equations of Navier-Stokes and Abstract Parabolic Equations, Vieweg, Braunschweig 1985.
[5] R. Temam, Infinite-Dimensional Dynamical Systems, in Mechanics and Physics, Springer, New York 1988.
[6] E. Hopf, Math. Nachr. **4**, 213 (1951).
[7] D. K. Arrowsmith and C. M. Place, An Introduction to Dynamical Systems, Cambridge University Press, Cambridge 1990, Chapter 1.
[8] D. Ruelle, Elements of Differentiable Dynamics and Bifurcation Theory, Academic Press, San Diego, Cal. 1989, Chapter 1.
[9] R. Temam, Navier Stokes Equations, North-Holland, Amsterdam 1977.
[10] V. Girault and P. A. Raviart, Finite Element Methods for Navier-Stokes Equations, Springer, Berlin 1986.
[11] R. S. Meyer-Spasche and H. B. Keller, J. Comput. Phys. **35**, 100 (1980).
[12] R. D. Moser, P. Moin, and A. Leonard, J. Comput. Phys. **52**, 524 (1983).
[13] P. S. Marcus, J. Fluid Mech. **146**, 45 (1984).
[14] P. S. Marcus, J. Fluid Mech. **146**, 65 (1984).
[15] G. P. King, Y. Li, W. Lee, H. L. Swinney, and P. S. Marcus, J. Fluid Mech. **141**, 365 (1984).
[16] D. C. Leslie, Developments in the Theory of Turbulence, Clarendon Press, Oxford 1973.
[17] G. Backus, Ann. Phys. **4**, 372 (1958).
[18] G. Backus, Rev. Geophys. **24**, 75 (1986).
[19] M. Nagata, J. Fluid Mech. **169**, 229 (1986).
[20] M. Nagata, J. Fluid Mech. **188**, 585 (1988).
[21] R. M. Clever and F. H. Busse, J. Fluid Mech. **198**, 345 (1989).
[22] B. J. Schmitt and W. V Wahl, Decomposition of solenoidal fields into poloidal fields, toroidal field and the mean flow. Application to the Boussinesq equations, to appear.
[23] G. Knorr, J. P. Lynov, and H. L. Pécseli, Z. Naturforsch. **45a**, 1059 (1990).
[24] B. Friedman, Principles and Techniques of Applied Mathematics, John Wiley, New York 1956.
[25] M. J. Lighthill, Introduction to Fourier Analysis and Generalized Functions, Cambridge University Press, Cambridge 1964, Chapter 5, Section (5.2).
[26] K. Yosida, Functional Analysis, Springer, Berlin 1978; the Friedrichs extension is discussed on p. 317.
[27] G. Ströhmer, to be published.
[28] S. Chandrasekhar and W. H. Reid, Proc. Nat. Acad. Sci. **43**, 521 (1957).
[29] S. Chandrasekhar, Hydrodynamic and Hydromagnetic Stability, Dover Publications, 1961.
[30] G. Ioss, J. Fluid Mech. **173**, 273 (1986).
[31] P. Chaussat, Y. Demay, and G. Ioss, Arch. Ration. Mech. Anal. **99**, 213 (1987).
[32] F. Serrin, Arch. Ration. Mech. Anal. **3**, 1 (1959).
[33] C. S. Yih, Arch. Ration. Mech. Anal. **47**, 288 (1972).
[34] W. H. Reid and D. L. Harris, Phys. Fluids **1**, 102 (1958).
[35] T. Gebhardt and S. Grossmann, Z. Naturforsch. **46a**, 669 (1991).
[36] F. Marqués, Phys. Fluids A **2**, 729 (1990).
[37] E. N. Lorenz, J. Atmos. Sci. **20**, 130 (1963).
[38] G. I. Taylor, Phil. Trans. Roy. Soc. London, Ser. A **233**, 289 (1923).
[39] P. R. Fenstermacher, H. L. Swinney, and J. P. Gollub, J. Fluid Mech. **94**, 103 (1979).